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Scalar tensor theories and variable rest masses

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Abstract. We consider a large class of scalar tensor theories which are derivable from a certain parametrised Lagrangian. The field equations are found and solved exactly for spherical symmetry. We identify the parameters appearing in this solution and use it to calculate the experimental predictions of this class of theories. This class contains a large variety of theories including general relativity, the Brans–Dicke theory, and theories with variable rest masses. Particular attention is paid to whether or not variable rest-mass effects are observable. We find that such effects are observable even if the theory is transformed back to laboratory units and cannot be ruled out by the standard solar system experiments.

1. Introduction

We investigate a broad class of scalar tensor theories in this paper. This class includes theories where particle rest masses vary with position (in some units) and therefore we learn something about the experimental consequences of such effects. A similar investigation has been carried out by Bekenstein (1977) for a different class of theories. The experimental consequences of various classes of scalar tensor theories have also been studied by Brans and Dicke (1961), Dicke (1962), Bergmann (1968), Wagoner (1970), Jordan (1948, 1955), Thirry (1948) and Nordtvedt (1970). Nutku (1969), Will (1974a) and Ni (1972) have discussed these theories within the PPN framework. The latter is not suitable for our purposes in general because we explicitly want to discuss theories which have variable particle masses.

The theories we want to consider are given by the general variational principle:

$$\delta \int (R\psi^{A} + \beta\psi^{i}\psi_{i}\psi^{A-2} + \kappa L\psi^{E})\sqrt{-g}\,\mathrm{d}^{4}x = 0. \tag{1}$$

Here A, E, β and κ are constants, R is the Ricci scalar, and ψ is the scalar field ($\psi_i \equiv \psi_{,i}$). ψ has dimension G^{-1} where G is the gravitational constant so that κ has different dimensions depending on the value of A and E. L is the Lagrangian for matter and includes particles, electromagnetic fields, etc, in general. For some uses below we will take L to be the particle Lagrangian

$$L_{\text{part.}} = \frac{2}{\sqrt{-g}}\sum_{p}\int m_p[-g_{ik}(x)\dot{z}^{i}\dot{z}^{k}]^{1/2}\delta^{4}(x-z_p)\,\mathrm{d}\tau_p \tag{2}$$

where $\dot{z}^{i} = \mathrm{d}z^{i}_{p}/\mathrm{d}\tau_p$ and τ_p is the path parameter of the pth particle. The variational principle (1) has been considered in a nice paper by Harrison (1972). (We will use Harrison's signature and sign conventions which are the same as the ones in Weinberg (1972). Note that Brans–Dicke use opposite signs for the Einstein tensor and opposite signs for $R^{\mu}{}_{\nu\alpha\beta}$ in terms of the Christoffel symbols. Also we will always define covariant derivatives in terms of the usual Christoffel symbols so that the metric is always covariantly constant.) Its advantages are that scalar-conformal transformations convert the theory into a theory which is still of the form (1). Equation (1) includes many theories besides the Brans–Dicke theory and Einstein's theory (if the appropriate energy momentum tensor for the scalar field is used), including non-metric theories with variable rest masses and theories which include creation. The functional dependence on ψ is explicit and simple, however.

Harrison (1972) showed that (1) is invariant under a scalar-conformal transformation of the form

$$\begin{aligned} &\psi \rightarrow \phi = \psi^{1/\mu}\chi \\ &g_{ik} \rightarrow g'_{ik} = \phi^{F} g_{ik} \\ &m(g_{ik}) \rightarrow m'(g'_{ik}) = \phi^{-H} m(g_{ik}) \text{ for particle masses} \end{aligned} \tag{3}$$

where χ is a constant and ϕ is dimensionless. (1) is invariant under (3) in the sense that (3) gives a new variational principle in terms of $g'_{i\kappa}$ and ϕ which is of the form (1) with analogous parameters A', E', κ' and β' given by

$$\begin{aligned} &A' = \mu A - F \\ &E' = \mu E + H - F/2 \\ &\beta' = \mu^{2}\beta + 3F(\mu A - F/2) \\ &\kappa' = \kappa\chi^{E'-A'} = \kappa\chi^{\mu(E-A)+H+F/2}. \end{aligned} \tag{4}$$

Note that because of (3), the matter Lagrangian (2) can be considered in general to have a hidden parameter H which determines how particle rest masses vary with position. Harrison primarily considered matter Lagrangians of the form (2) where

$$L(g_{i\kappa}) \rightarrow L'(g'_{i\kappa}) = \phi^{-3/2F-H} L(g_{i\kappa}).$$

If electromagnetic interactions are included, then we must have $F = 2H$ or this equation and hence (4) will not hold. We will return to this requirement below.

We will be interested in seeing which, if any, of the theories described by (1) agree with experiment, and in particular in seeing if $H \neq 0$ is compatible with experiment. (3) will be useful later in converting one theory into another and in particular in converting a theory with $H = 0$ into one with $H \neq 0$.

Note that the parameter space of (3) includes the parameters for the type of units transformation considered by Dicke (1962) if we take $\mu = F = 2H = 1$ and $\chi = \psi'^{-1/\mu}$ where ψ' is the transformed version of ψ (assumed by Dicke to be a constant) and ϕ is a dimensionless x_{μ}-dependent scale factor.

We will solve the field equations arising from (1) for general A, E, β, κ and H exactly for a spherically symmetric point mass in § 2 below. In § 3, we determine the equations of motion for a test particle including the effects of H and use them to determine the integration constants in the exact solution in terms of the mass M of the central body and the gravitational constant G as measured in a local Cavendish experiment. In § 4, we work out the physical predictions, for theories based on (1), for the Eötvös *et al* (1922) experiment and for the three classic tests of general relativity. Finally, we discuss the physical viability and observability of $H \neq 0$ effects for theories based on (1) in § 5.

2. Spherically symmetric solution

We now want to find the exact spherically symmetric solution to the field equations. In the rest of this paper, ψ will denote the general scalar field both before (3) has been carried out and after, so that the parameter H appears in (1) in the general case. Independent variations of g_{ik} and ψ in (1) give the field equations (Harrison 1972)

$$(R^i_k - \tfrac{1}{2}\delta^i_k R)\psi^A + D^i_k(\psi) + \kappa T^i_k \psi^E = 0 \tag{5}$$

and

$$AR\psi^A - \beta[2\psi\Box\psi - (2-A)\psi^i\psi_i]\psi^{A-2} + \kappa EL\psi^E = 0 \tag{6}$$

where

$$D^i_k(\psi) \equiv (\psi^A)^{;i}_{;k} - \delta^i_k \Box\psi^A + \beta(\psi^i\psi_k - \tfrac{1}{2}\delta^i_k\psi^i\psi_i)\psi^{A-2}. \tag{7}$$

Contracting (5) and substituting for R in (6) simplifies (6) to

$$\left(3 + \frac{2\beta}{A^2}\right)\Box\psi^A - \kappa T\psi^E\left(1 + \frac{2E}{A}\right) = 0 \qquad (A \neq 0) \tag{8}$$

or to

$$\beta\psi^{-1}\Box\psi - \beta\psi^i\psi_i\psi^{-2} - \kappa ET\psi^E = 0 \qquad (A = 0) \tag{9}$$

where $L = 2T$ from the pressureless particle Lagrangian (2).

The exact vacuum solution to (5) and (8) in isotropic coordinates is now

$$-g_{00} = Q_0\left|\frac{1 - B/r}{1 + B/r}\right|^{2/\lambda} \tag{10}$$

$$g_{11} = J_0\left(1 + \frac{B}{r}\right)^4\left(\frac{1 - B/r}{1 + B/r}\right)^{2(\lambda - C - 1)/\lambda} \tag{11}$$

$$\psi = \psi_0\left(\frac{1 - B/r}{1 + B/r}\right)^{C/A\lambda} \tag{12}$$

where

$$\lambda \equiv \left[(C+1)^2 - C\left(1 - \frac{\beta C}{2A^2}\right)\right]^{1/2} \tag{13}$$

and Q_0, J_0, ψ_0, C and B are arbitrary constants restricted by $(C+1)^2 - C(1 - \beta C/2A^2) > 0$. Boundary conditions at ∞ give $Q_0 = J_0 = 1$ leaving three constants to be determined. For the case $A = 0$, the above expressions (10) to (13) are still valid if we let $C \to 0$ and $C/A \to \eta$ where now, for example,

$$\lambda \equiv (1 + \tfrac{1}{2}\beta\eta^2)^{1/2} \qquad (A = 0 \text{ case}). \tag{14}$$

In this case ψ_0, η and B are the constants to be determined.

The above solution of our field equations reduces to the Misner solution of the Brans–Dicke (1961) equations for $A = 1$ as it should. The solution (10) to (13) can be obtained either by laborious calculation or by noticing that our field equations for the quantity ψ^A have the same form as the Brans–Dicke equations if their coupling constant $\omega \to \beta/A^2$. Thus our solution for general A is a simple transformation of the Misner solution. Also note that other solutions where C is restricted differently as discussed by Brans (1962) can be transformed similarly. We are most interested in the solution (10) to (14).

3. Equations of motion and solution parameters

We now want to determine the parameters C (or η), B and ψ_0 in our exact solution. First, however, we need the equations of motion of an uncharged test particle. If we vary (1) with respect to the matter variables in L where L is given by (2) with a variable rest mass $m(x)=\psi^{-H}m_0$, we find for a particular particle (dropping p subscripts) that

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau}+\{^{\alpha}_{\pi l}\}u^\pi u^l+\frac{m_{,\beta}}{m}\{u^\alpha u^\beta+g^{\alpha\beta}\}=0 \tag{15}$$

where $u^\alpha\equiv \mathrm{d}x^\alpha/\mathrm{d}\tau$. Putting in our explicit dependence of m on the scalar field ψ then gives

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau}+\{^{\alpha}_{\pi l}\}u^\pi u^l-\frac{H\psi_\beta}{\psi}\{u^\alpha u^\beta+g^{\alpha\beta}\}=0. \tag{16}$$

Thus test particles follow geodesics if and only if $H=0$.

We are now ready to determine C, B and ψ_0 in terms of A, E, H, β, κ, M and G where M is the central mass and G is the gravitational constant as measured in a local Cavendish experiment. We will discuss $A\neq 0$ first and then the $A=0$ case. We will need three equations. The first will identify G from the equations of motion for a radially infalling particle. The other two will arise from comparing the linearised g_{00} equation and linearised ψ equation (with a central mass present in $T_{\alpha\beta}$) with the exact vacuum solution (10) through (13).

For a radially accelerating particle, we can let $u^1=u^2=u^3=0$ instantaneously. Then the equations of motion (16) become

$$\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}=\tfrac{1}{2}g'_{00}g^{11}(u^0)^2+H\frac{\psi'}{\psi}g^{11} \tag{17}$$

where a prime denotes a derivative with respect to r. To lowest order $g^{11}\approx 1$ and $(u^0)^2\approx 1$. Substituting g'_{00} and ψ' from our exact solution then gives

$$\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}=\frac{-2B}{\lambda r^2}\left(1-\frac{HC}{A}\right)\qquad (A\neq 0). \tag{18}$$

We can write this as

$$r\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}=-\frac{2B}{\lambda r}\left(1-\frac{HC}{A}\right)\qquad (A\neq 0) \tag{18'}$$

where now the left-hand side is dimensionless. The left-hand side of this equation can be measured to lowest order and will not depend on the units or coordinate system used, i.e. whether laboratory units with m = constant are used or some other units. *To lowest order* the right-hand side can then be defined in these units to be $-GM/r$ where this G is then the gravitational constant as measured to lowest order in a local Cavendish experiment. Thus, we can make the identification

$$G=\frac{2B}{\lambda M}\left(1-\frac{HC}{A}\right)\qquad (A\neq 0). \tag{19}$$

For $A=0$ we can merely let $C/A\to\eta$ here. A double check that (19) is correct is provided by the gravitational red-shift experiment (see (39) and the discussion following it below). For B/λ given by (19), the red-shift experiment predicts the same result for our general scalar tensor theory as for general relativity, thus establishing that local Lorentz frames fall with the same acceleration as test particles as one might expect. Any change in (19) would lead to a discrepancy between the acceleration of local Lorentz frames and of test particles.

Now let us turn to the linearised equation for g_{00}. The $k=0, j=0$ component of (5) can be written as

$$R_{00}\psi^{A} = -\kappa\psi^{E}\left[T_{00} - \frac{g_{00}T(1+\beta/A^2 - E/A)}{(3+2\beta/A^2)}\right] \qquad (A\neq 0), (3+2\beta/A^2 \neq 0). \tag{20}$$

Now to lowest non-trivial order we have (Weinberg 1972)

$$R_{00}^{(2)} = \tfrac{1}{2}\nabla^2 g_{00}^{(2)} \tag{21}$$

so that

$$\nabla^2 g_{00}^{(2)} = -\kappa\psi_0^{E-A}T_{00}^{(0)}\left(\frac{4+2\beta/A^2+2E/A}{3+2\beta/A^2}\right). \tag{22}$$

Now $T_{00}^{(0)} \approx -T \approx \delta^3(x)M$ so the retarded time solution to (22) can be immediately written down. When this solution is compared with $g_{00}^{(2)} = 4B/\lambda r$ from (10), we can identify

$$\frac{4B}{\lambda} = M\frac{\kappa}{4\pi}\psi_0^{E-A}\left(4+\frac{2\beta}{A^2}+\frac{2E}{A}\right)\left(3+\frac{2\beta}{A^2}\right)^{-1} \qquad (A\neq 0). \tag{23}$$

A similar calculation for $A=0$ yields

$$\frac{4B}{\lambda} = M\frac{\kappa}{4\pi}\psi_0^{E} \qquad (A=0). \tag{23'}$$

Similarly, the linearised equation for $\psi \equiv \psi_0 + \xi$, where ξ is small, from (8) is

$$\Box\xi = \frac{\kappa T(1+2E/A)\psi_0^{E-A+1}}{A(3+2\beta/A^2)} \qquad (A\neq 0), \qquad (3+2\beta/A^2\neq 0). \tag{24}$$

Putting a point mass into T as above, finding the retarded time solution, and comparing with $\psi \approx \psi_0 - (2B/r)\psi_0(C/A\lambda)$ from (12) gives finally

$$\frac{-2BC}{A\lambda} = \frac{1}{4\pi}\frac{\kappa M(1+2E/A)\psi_0^{E-A}}{A(3+2\beta/A^2)} \qquad (A\neq 0). \tag{25}$$

For $A=0$ from (9) we have by a similar calculation

$$\frac{-2B\eta}{\lambda} = \frac{1}{4\pi}\frac{\kappa M E\psi_0^{E}}{\beta} \qquad (A=0), \qquad (E\neq 0). \tag{26}$$

Note that $E\neq 0$ here. The $A=0$, $E=0$ case will be considered separately below.

Now we can use our three equations (19), (23) and (25) to determine the three integration constants C, B and ψ_0 in our exact solution. This gives for $A\neq 0$

$$C = \frac{-(1+2E/A)}{(2+\beta/A^2+E/A)} \tag{27}$$

$$\psi_0^{E-A} = \left(3+\frac{2\beta}{A^2}\right)G\left(\frac{4\pi}{\kappa}\right)\left[2+\frac{\beta}{A^2}+\frac{E}{A}+\frac{H}{A}\left(1+\frac{2E}{A}\right)\right]^{-1} \tag{28}$$

and

$$B = \lambda \frac{GM}{2(1-HC/A)} \tag{29}$$

where λ is given by (13) and C by (27). (28) reduces to

$$\psi_0^{-1} = \frac{G(3+2\beta)}{(4+2\beta)} \tag{30}$$

for $A = 1$, $E = 0$, $H = 0$, $\kappa = 8\pi$ which is the result obtained for this case by Brans and Dicke (1961). Note that κ is dimensionless for this choice of A and E. For $A = 0$ we can determine η, B and ψ_0 from the $A = 0$ version of (19), from (23′), and from (26). If $E \neq 0$ we get

$$\eta = \frac{-2E}{\beta} \qquad (A = 0, E \neq 0) \tag{31}$$

$$\psi_0^E = \frac{G(8\pi/\kappa)}{(1+2HE/\beta)} \qquad (A = 0, E \neq 0) \tag{32}$$

$$B = \lambda \frac{MG}{2(1+2HE/\beta)} \qquad (A = 0, E \neq 0) \tag{33}$$

where λ is given by (14) and (31). This completely determines our solution parameters except for the important case $A = 0$, $E = 0$ which is relevant to the Dicke (1962) theory. In this case, and only in this case, ψ uncouples from T in (9) and the linearised ψ equation no longer depends on M, so that a condition like (26) is no longer obtainable. ψ also contributes to the $g_{\alpha\beta}$ field equations only to higher order. Since ψ_0 does not now appear in the G equation or in the linearised g_{00} equation, these two equations can be solved for η and B and yield

$$\eta = \frac{1-8\pi G/\kappa}{H} \qquad (A = 0, E = 0) \tag{34}$$

$$B = \frac{\kappa}{8\pi} \frac{M}{2} [1+2\beta(1-8\pi G/\kappa)^2]^{1/2} \qquad (A = 0, E = 0). \tag{35}$$

Experimental results will be found to depend on C (or η). Expressions for C and η in (27) and (31) are independent of κ, but η in (31) (the $A = 0$, $E = 0$ case) depends on κ, and κ assumes somewhat the role of ψ_0 for this special case. In fact, for this case, Dicke (1962) chooses $\kappa = 8\pi\psi_0^{-1}$ where ψ_0 is the zeroth-order scalar field from the related Brans–Dicke (1961) theory. Using (30) and (34) then gives

$$\eta = \frac{-2}{\mu(3+2\beta)} \qquad (A = 0, E = 0) \tag{34'}$$

for this special case.

This concludes our discussion of the exact solution of the field equations, the equations of motion, and the determination of the integration constants. We are now in a position to calculate physical predictions for experiments for any theory described by (1).

4. Experimental predictions of the theories

We will now discuss the Eötvös experiment and the three classic tests of general relativity and will calculate the predictions of our theories for these tests in terms of the parameters contained in (1). Note that the PPN formalism (Will 1974a, Will and Nordtvedt 1972) cannot be applied and so detailed calculations must be done, because we have non-metric theories for some values of A, E and H in (1), and $H \neq 0$ effects are of interest to us.

4.1. Eötvös experiment

The Eötvös *et al* (1922) experiment and its modern extensions by Roll *et al* (1964) and Braginsky and Panov (1971) verifies to high accuracy the universality of free fall (UFF). UFF states that the world line of a freely falling test body is independent of its composition or structure. For our theories, taking the divergence of (5) yields

$$(T^{\beta}_{\alpha}\psi^{E})_{;\beta} + T\psi^{E}_{,\alpha} = 0. \tag{36}$$

If L in (1) contains electromagnetic contributions then (36) shows that Maxwell's equations will be modified for some values of A, E and H. It is then easy to see from calculations similar to those done in the T, H, e, μ formalism (Lightman and Lee 1973, Haugan and Will 1977) that charged particles will free fall differently from neutral ones and UFF will be violated. If we set $E=0$ and $F=2H$ so that $E'=0$ in (4), then no difficulties with the Eötvös experiment arise. We will return to these conditions imposed by the Eötvös experiment in § 5, meanwhile letting E, F and H remain unfixed. Notice that the $F=2H$ condition is also the requirement imposed if electromagnetic source terms are to be put consistently into L matter in Harrison's work.

4.2. Gravitational red shift

Gravitational red shift experiments (Pound and Rebka 1960, Pound and Snyder 1964) establish that local Lorentz frames fall with the same acceleration as test particles (Will 1974b), and so are meaningful. To derive the red shift for the general case in which the rest mass of particles can be a function of position, consider a transmitter at r_1 emitting frequency ν_1 and a second transmitter at a higher elevation r_2 which emits frequency ν_2. At r_2 we have a receiver which compares the received signal from the two transmitters, ν_{received_1} and $\nu_{\text{received}_2} \equiv \nu_2$. We want to find $(\nu_{\text{received}_1} - \nu_{\text{received}_2})/(\nu_{\text{received}_2})$. Now the frequencies of atomic transitions are proportional to the mass of the electron. Thus we have 'real' changes in frequencies given by

$$\frac{\nu_2}{\nu_1} = \frac{m_2}{m_1} = \frac{\psi_2^{-H}}{\psi_1^{-H}} = 1 + GM\frac{HC}{A}\frac{1}{(1-HC/A)}\left(\frac{1}{r_2} - \frac{1}{r_1}\right), \tag{37}$$

where we used (12) in linearised form and substituted B/λ from (29). We also get a gravitational red shift

$$\frac{\nu_{\text{received}_1} - \nu_1}{\nu_1} = \frac{-GM}{(1-HC/A)}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{38}$$

using g_{00} from (10) and again substituting B/λ from (29). Combining (37) and (38) with

$\nu_{\text{received}_2} \equiv \nu_2$ then gives

$$\frac{\nu_{\text{received}_1} - \nu_{\text{received}_2}}{\nu_{\text{received}_2}} = -GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{39}$$

Thus the 'real' energy level changes just compensate rather miraculously the H-dependent terms in the gravitational red shift to yield the usual Einstein gravitational red shift. In fact the second-order term in (39) is $(G^2M^2/2)(1/r_1 - 1/r_2)^2$ so that H-dependent terms cancel even in higher order! The $A = 0$ case goes through in the same way. Thus *all* theories described by (1) have the usual gravitational red shift. This rather interesting result is not obvious since we are including some non-metric theories.

4.3. *Deflection of starlight*

Light travels a null geodesic even in the general case where particle masses are functions of position. Thus the metric alone determines the deflection of starlight. If we parametrise the isotropic metric as

$$\begin{aligned} ds^2 = &-\left(1 - 2\alpha\frac{MG}{r} + 2\delta\frac{M^2G^2}{r^2} + \ldots\right) dt^2 \\ &+ \left(1 + 2\gamma\frac{MG}{r} + \ldots\right)(d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2), \end{aligned} \tag{40}$$

then the deflection of starlight is

$$\Delta\phi_{\text{light}} = \frac{4MG}{r_0}\left(\frac{\alpha + \gamma}{2}\right) \tag{41}$$

where general relativity predicts $4MG/r_0$. Expanding our exact solution (10) and (11) and identifying α and γ gives then

$$\alpha = 1/\xi \qquad \delta = 1/\xi^2 \qquad \gamma = (C+1)/\xi \tag{42}$$

where $\xi \equiv 1 - HC/A$. Thus we have

$$\Delta\phi_{\text{light}} = \frac{4MG}{r_0}\left(\frac{1 + C/2}{1 - HC/A}\right) \qquad (A \neq 0) \tag{43}$$

where C is given by (27). This depends on H, A, E and β now. For $H = 0, E = 0, A = 1$ the correction factor becomes $(3 + 2\beta)(4 + 2\beta)^{-1}$ which is the Brans–Dicke result. For the $A = 0$ case, the correction factor becomes $(1 - H\eta)^{-1}$ where η is given by (31) if $E \neq 0$ or by (34′) if $E = 0$.

4.4. *Advance of perihelion of Mercury*

Our modified equations of motion now also play a role. It is most convenient to work in standard rather than in isotropic coordinates now. In standard coordinates, we can write

$$\begin{aligned} d\tau^2 &= P(r)\, dt^2 - N(r)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \\ \psi^{-2H}(r) &= \psi_0^{-2H}\left(1 + \frac{VGM}{r} + \frac{WG^2M^2}{r^2}\right) \end{aligned} \tag{44}$$

where τ is proper time and

$$P(r) \equiv 1 - 2\alpha \frac{MG}{r} + 2(\delta - \alpha\gamma)\frac{M^2G^2}{r^2}$$

$$N(r) \equiv 1 + 2\gamma \frac{MG}{r}$$

$$V \equiv \frac{2HC}{A}\left(1 - \frac{HC}{A}\right)^{-1}$$

$$W \equiv 2\frac{HC}{A}\left(1 + \frac{HC}{A} + C\right)\left(1 - \frac{HC}{A}\right)^{-2}. \tag{45}$$

α, δ and γ are given in (42). Manipulating and integrating our equations of motion (16) along the lines of the calculation by Weinberg (1972) leads to an orbit of the form

$$\phi = \int N(r)^{1/2}\,\mathrm{d}r\, r^{-2}\left(\frac{1}{J^2P(r)} - \frac{E}{J^2} - \frac{1}{r^2} + \frac{1 - (\psi/\psi_0)^{-2H}}{J^2}\right)^{-1/2} \tag{46}$$

where E and J are integration constants related to the conserved energy and angular momentum. Again, following Weinberg, this leads to a perihelion shift per revolution of

$$\Delta\phi_{\text{perihelion}} = \frac{6\pi MG}{L}\left(\frac{2\alpha^2 - \delta + \alpha\gamma + \gamma - W/2}{3}\right) \tag{47}$$

or finally to

$$\Delta\phi_{\text{perihelion}} = \frac{6\pi MG}{L}(3 + HC/A + 2C)\tfrac{1}{3}(1 - HC/A)^{-1} \qquad (A \neq 0) \tag{48}$$

where L is the semilatus rectum of the ellipse. The first factor in (48) is the prediction of general relativity. For $A = 1$, $E = 0$ and $H = 0$ this reduces to the correct Brans–Dicke result. We note in (47) that the final result depends only on the second-order terms W in $\psi(r)^{-2H}$ in standard coordinates and not on the first-order terms V in (44). Of course, first-order terms in the expansion of (12) contribute, since (12) is in isotropic coordinates. g_{00} behaves similarly.

If we repeat the calculation for $A = 0$, we can write the result as

$$\Delta\phi_{\text{perihelion}} = \frac{6\pi MG}{L}\left(\frac{3 + H\eta}{3(1 - H\eta)}\right) \qquad (A = 0). \tag{49}$$

5. Discussion

Let us take $E = 0$ in (1) and $F = 2H$ in (4). The Eötvös experiment is then satisfied and electromagnetic fields can be put into Harrison's work consistently. The red-shift experiment also predicts the same results as general relativity for all of these scalar tensor theories. From (27) and (43)

$$\Delta\phi_{\text{light}} = (\text{general relativity result})\left(\frac{3A^2 + 2\beta}{4A^2 + 2\beta + 2HA}\right) \qquad (A \neq 0). \tag{50}$$

The $A=0$, $E=0$ case is essentially equivalent to the Dicke (1962) theory and will not be discussed further. From (27) and (48), the advance of perihelion is given for $E=0$ by

$$\Delta\phi_{\text{perihelion}} = (\text{general relativity result})\left(\frac{4A^2+3\beta-HA}{6A^2+3\beta+3HA}\right) \qquad (A\neq 0). \tag{51}$$

If $H=0$ and $A=1$, (50) and (51) reduce properly to the Brans–Dicke (1961) result, with our β equal to their ω parameter. We obtain the general relativity limit as $\beta\to\infty$. Experimentally the correction bracket in (50) is $1{\cdot}015\pm0{\cdot}011$ as found by Fomalont and Sramek (1975). The correction bracket in (51), as reduced by Shapiro *et al* (1976) under the assumption that the solar quadrupole is exclusively due to uniform rotation, is $1{\cdot}003\pm0{\cdot}005$.

The results in (50) and (51) represent the predictions of a theory given by the Lagrangian (1) with $H\neq0$ describing variable particle masses through (2) and (3). Since experiments are usually carried out in laboratory units which are *defined* with particle masses constant, it is useful to transform (50) and (51) to these units with a units transformation. A true units transformation is of the form (3) and (4) carried out in such a way that the numerical value of the Lagrangian (1) is unchanged. Note that (4) relates theories with parameters A, E and constant particle masses to theories with parameters A', E' and variable particle masses. In order to use (4) to transform to laboratory units, then, we must change notation in (50) and (51), replacing A and β with A' and β', since (50) and (51) refer to variable rest masses in general. (We wrote equations (5)–(51) without primes to avoid hundreds of primes.) This gives

$$\Delta\phi_{\text{light}} = (\text{GR result})\left(\frac{3A'^2+2\beta'}{4A'^2+2\beta'+2HA'}\right) \text{ (units with variable particle masses)} \tag{50'}$$

and

$$\Delta\phi_{\text{perihelion}} = (\text{GR result})\left(\frac{4A'^2+3\beta'-HA'}{6A'^2+3\beta'+3HA'}\right) \text{ (units with variable particle masses).} \tag{51'}$$

Using (4) with $F=2H$ and $E=E'=0$ then allows us to write (50′) and (51′) in terms of A_l and β_l, where a subscript l has been added to denote laboratory units to prevent confusion with A and β in (50) and (51). Thus $A'=\mu A_l-2H$ and $\beta'=\mu^2\beta_l+6H(\mu A_l-H)$ from (4). This gives

$$\Delta\phi_{\text{light}} = (\text{GR result})\left(\frac{3A_l^2+2\beta_l}{4A_l^2+2\beta_l-(2H/\mu)A_l}\right) \text{ (laboratory units)} \tag{52}$$

and

$$\Delta\phi_{\text{perihelion}} = (\text{GR result})\left(\frac{4A_l^2+3\beta_l+(H/\mu)A_l}{6A_l^2+3\beta_l+3\beta_l-3(H/\mu)A_l}\right) \text{ (laboratory units).} \tag{53}$$

(The scalar transformation parameter μ can be set equal to 1 without real loss of generality). We notice the very important result now that, even after we have transformed a theory with variable particle masses back to laboratory units, the original non-zero value of H still affects experiments and can be measured. Thus variable particle masses have observable effects. In addition, A_l, β_l and H give us considerable freedom in fitting experiment to theory. (52) and (53) can easily be made compatible with the experimental results with reasonable choice of the parameters.

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